Two point boundary value problems for ordinary differential equations, uniqueness implies existence

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In a seminal paper, Lasota and Opial (Colloq. Math., 1967) proved that for second order ordinary differential equations, global existence and uniqueness of solutions of initial value problems and uniqueness of solutions of two point conjugate (Dirichlet) boundary value problems implies existence of solutions of two point conjugate boundary value problems.

Following this work many related results were obtained and principal contributors include Jackson and his students, Hartman, and Henderson. Henderson and Eloe recently gathered these results in the monograph Nonlinear Interpolation and Boundary Value Problems (EH, 2016).

In the context of the second order ordinary differential equation, if the nonlinear term is monotone increasing with respect to position, the solutions of two point conjugate boundary value problems are unique if they exist.

If the difference, u, of distinct solutions has an absolute extreme point at c in the interior of the interval, then the sign of $u''(c)$ given by the second derivative test is opposite the sign of $u''(c)$ given by the monotonicity.

Let $n \geq 2$ denote an integer and let $a < T_1 < T_2 < b$. Let $a_i \in \mathbb{R}$, $i = 1, \dots, n$. We shall consider the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t), \dots, y^{(n-1)}(t)), \quad t \in [T_1, T_2], \tag{1}$$

where $f : (a, b) \times \mathbb{R}^n \to \mathbb{R}$, or the ordinary differential equation

$$y^{(n)}(t) = f(t, y(t)), \quad t \in [T_1, T_2], \tag{2}$$

where $f : (a, b) \times \mathbb{R} \to \mathbb{R}$. We shall prove existence of solutions of two-point boundary value problems for either (1) or (2) with the boundary conditions

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-1, \quad y(T_2) = a_n, \tag{3}$$

under the assumption of uniqueness of solutions of boundary value problems for either (1) or (2) with the boundary conditions, for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \dots, n-1, \quad y^{(j-1)}(T_2) = a_n. \tag{4}$$

There are two contributions of this work.

The first is to obtain that the uniqueness of solutions of the boundary value problems (1), (4) implies the existence of solutions of the boundary value problems (1), (3).

The second is to obtain a verifiable hypothesis with respect to monotonicity of f such that solutions of (2), (4) are unique if they exist.

We will begin with the second contribution.

Theorem (monotone increasing implies uniqueness of solutions)

Assume that $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ *is continuous and assume there exists a positive constant,* P, *such that*

$$|f(t, y) - f(t, z)| \leq P|y - z|$$

for all $(t, y), (t, z) \in (a, b) \times \mathbb{R}$. *Assume in addition that* $\frac{\partial f}{\partial y} = f_y$ *exists and*

$$\frac{\partial f}{\partial y} = f_y : (a, b) \times \mathbb{R} \to (0, \infty).$$

Then solutions of the boundary value problem (2), (4) *are unique if they exist.*

Let $j \in \{1, 2\}$. Assume that y_1 and y_2 are distinct solutions of the boundary value problem (2), (4). We first argue that there exists $T_3 \in (T_1, T_2)$ such that $(y_1 - y_2)(T_3) = 0$. So, for the sake of contradiction, assume $y_1 - y_2$ is of constant sign on (T_1, T_2) and without loss of generality assume $(y_1 - y_2)(t) > 0$ for $T_1 < t < T_2$. Set $u(t) = (y_1 - y_2)(t)$. Then

$$u^{(n)}(t) = f(t, y_1(t)) - f(t, y_2(t)) > 0, \quad T_1 < t < T_2,$$

which implies $u^{(n-1)}(t)$ is increasing on (T_1, T_2). Since $u^{(i-1)}(T_1) = u^{(j-1)}(T_2) = 0$, $i = 1, \ldots, n-1$, repeated applications of Rolle's theorem implies $u^{(n-1)}(t)$ has a root in (T_1, T_2).

With $u^{(n-1)}(t)$ is increasing on (T_1, T_2) and $u^{(n-1)}(t)$ has a root in (T_1, T_2) conclude that $u^{(n-1)}(t)$ has precisely one root in (T_1, T_2) and thus, $u^{(n-1)}(T_1) < 0$.

So u satisfies

$$u^{(i-1)}(T_1) = 0, \quad i = 1, \ldots, n-1, \quad u^{(n-1)}(T_1) < 0,$$

and Taylor's theorem implies that $u(t) < 0$ in a right neighborhood of $t = T_1$. This contradicts that $(y_1 - y_2)(t) > 0$ for $T_1 < t < T_2$. Thus, there exists $T_3 \in (T_1, T_2)$ such that $(y_1 - y_2)(T_3) = 0$.

Let

$$S = \{t \in (T_1, T_2) : (y_1 - y_2)(t) = 0\}.$$

We have just shown $S \neq \emptyset$. Let $\tau = \inf S$. If $\tau > T_1$, argue that $(y_1 - y_2)(\tau) = 0$. This follows by continuity if τ is a limit point of S and by definition if τ is an isolated point of S. Thus if $\tau > T_1$, y_1 and y_2 are distinct solutions of a boundary value problem (2), (4) for $T_2 = \tau$. Now apply the argument in the preceding paragraph and show there exists $T_3 \in (T_1, \tau)$ such that $(y_1 - y_2)(T_3) = 0$; in particular, the assumption that $\tau = \inf S > T_1$ is false.
So, $\inf S = T_1$. Find $T \in S$ such that $0 < T - T_1 < \delta = \frac{1}{2P}$. Then an application of the contraction mapping principle, not developed in this presentation, implies $y_1 \equiv y_2$ on $[T_1, T]$. Now Condition (B) implies $y_1 \equiv y_2$ on (a, b).

We now return to the first contribution.

$$y^{(n)}(t) = f(t, y(t), \ldots, y^{(n-1)}(t)), \quad t \in [T_1, T_2],$$

for $j \in \{1, 2\}$,

$$y^{(i-1)}(T_1) = a_i, \quad i = 1, \ldots, n-1, \quad y^{(j-1)}(T_2) = a_n.$$

We also have need to state the $n-$point conjugate boundary conditions

$$y(t_i) = a_i, \quad i = 1, \ldots, n, \tag{5}$$

where $a < t_1 < t_2 < \cdots < t_n < b$, and $a_i \in \mathbb{R}, i \in \{1, \ldots, n\}$.

With respect to (1) common assumptions for the types of results that we consider are:

(A) $f(t, y_1, \ldots, y_n) : (a, b) \times \mathbb{R}^n \to \mathbb{R}$ is continuous;

(B) Solutions of initial value problems for (1) are unique and extend to (a, b);

(C) There exists at most one solution of each $n-$point conjugate boundary value problem (1), (5) on (a, b).

With respect to (2) the assumptions (A) and (B) are replaced, respectively, by

(A') $f(t, y) : (a, b) \times \mathbb{R} \to \mathbb{R}$ is continuous;

(B') Solutions of initial value problems for (2) are unique and extend to (a, b).

We shall replace Condition (C), (that is, $n-$point disconjugacy), with a Condition (D), stated here.

(D) Solutions of the two-point boundary value problems (1), (4) are unique if they exist.

We shall then modify the sequential compactness argument of Lasalle and Opial to obtain existence of solutions of each member of the family of two-point boundary value problems (1), (3).

A generalized mean value theorem

Set $h > 0$ and choose $t_0 = T, t_1 = T + h, \ldots, t_i = T + ih$ to be equally spaced. If a function z is i times continuously differentiable on $[T, T + ih]$ then there exists $c \in (T, T + ih)$ such that

$$\frac{\sum_{l=0}^{i}(-1)^{i-l}\binom{i}{l}z(T + ih)}{h^i} = z^{(i)}(c). \tag{6}$$

For example, if $i = 1$, (6) is the mean value theorem and if $i = 2$, there exists $c \in (T, T + 2h)$ such that

$$\frac{z(T) - 2z(T + h) + z(T + 2h)}{h^2} = z''(c).$$

Continuous dependence on initial conditions:

Lemma

Assume that with respect to (1)*, Conditions (A) and (B) are satisfied. Then, given a solution y of* (1)*, given $t_0 \in (a, b)$, given any compact interval $[c, d] \subset (a, b)$, and given $\epsilon > 0$, there exists $\delta > 0$ such that if z is a solution of* (1) *satisfying $|y^{(i-1)}(t_0) - z^{(i-1)}(t_0)| < \delta$, $i = 1, \ldots, n$, then $|y^{(i-1)}(t) - z^{(i-1)}(t)| < \epsilon$, $i = 1, \ldots, n$, for all $t \in [c, d]$.*

Continuous dependence on boundary conditions:

Theorem

Assume that with respect to (1) *Conditions (A), (B), and (D) are satisfied. Let* $j \in \{1, 2\}$.

(i) *Given any* $a < T_1 < T_2 < b$, *and any solution* y *of* (1), *there exists* $\epsilon > 0$ *such that if* $|T_{11} - T_1| < \epsilon$, $|y^{(i-1)}(T_1) - y_{i1}| < \epsilon$, $i = 1, \ldots, n-1$, *and* $|T_{21} - T_2| < \epsilon$, $|y^{(j-1)}(T_2) - y_{n1}| < \epsilon$, *then there exists a solution* z *of* (1) *such that* $z^{(i-1)}(T_{11}) = y_{l1}$, $i = 1, \ldots, n-1$, $z^{(j-1)}(T_{21}) = y_{n1}$.

(ii) *If* $T_{1k} \to T_1$, $T_{2k} \to T_2$, $y_{ik} \to y_i$, $i = 1, \ldots, n$ *and* z_k *is a sequence of solutions of* (1) *satisfying* $z_k^{(i-1)}(T_{1k}) = y_{ik}$, $i = 1, \ldots, n-1$, $z_k^{(j-1)}(T_{2k}) = y_{nk}$, *then for each* $i \in \{1, \ldots, n\}$, $z_k^{(i-1)}$ *converges uniformly to* $y^{(i-1)}$ *on compact subintervals of* (a, b).

Theorem

Assume that with respect to (1), *Conditions (A), (B), and (D) are satisfied. Then for each* $a < T_1 < T_2 < b$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, *the two point boundary value problem* (1), (3) *has a solution.*

PROOF: This is a shooting method. Let $m \in \mathbb{R}$ and denote by $y(t; m)$ the solution of the initial value problem (1), with initial conditions

$$y^{(i-1)}(T_1; m) = a_i, \quad i = 1, \ldots, n-1, \quad y^{(n-1)}(T_1; m) = m.$$

Let

$$\Omega = \{p \in \mathbb{R} : \text{ there exists } m \in \mathbb{R} \text{ with } y(T_2; m) = p\}.$$

So the theorem is proved by showing $\Omega = \mathbb{R}$. By Condition (B), $\Omega \neq \emptyset$, so the theorem is proved by showing Ω is open and closed. That Ω is open follows from continuous dependence on boundary conditions.

To show Ω is closed, let p_0 denote a limit point of Ω and without loss of generality let p_k denote a strictly increasing sequence of reals in Ω converging to p_0. Assume $y(T_2; m_k) = p_k$ for each $k \in \mathbb{N}_1$. It follows by the uniqueness of solutions, Condition (D), that

$$y^{(j-1)}(t; m_{k_1}) \neq y^{(j-1)}(t; m_{k_2}), \quad t \in (T_1, b), \tag{7}$$

for each $j \in \{1, 2\}$, if $k_1 < k_2$ and in particular,

$$y(t; m_1) < y(t; m_k) \quad t \in (T_1, b), \tag{8}$$

for each k.

Either $y'(T_2; m_k) \leq 0$ infinitely often or $y'(T_2; m_k) \geq 0$ infinitely often. Relabel if necessary and assume $y'(T_2; m_k) \leq 0$ or $y'(T_2; m_k) \geq 0$ for each k. Finally note that (7) implies that we may assume $y'(T_2; m_k) < 0$ or $y'(T_2; m_k) > 0$ for each k.
We first assume the case $y'(T_2; m_k) < 0$ for each k. Find $T_2 < T_3 < b$ such that $y'(t; m_1) \leq 0$, for $t \in [T_2, T_3]$. Then $y(t; m_1)$ is decreasing on $[T_2, T_3]$. By (8), if $t \in [T_2, T_3]$ and $k \geq 1$, then

$$L = y(T_3; m_1) \leq y(t; m_1) \leq y(t; m_k). \tag{9}$$

Fix k and find $T_2 < T_{3k} \leq T_3$ such that $y'(t; m_k) < 0$ on $[T_2, T_{3k}]$. Then $y(t; m_k)$ is decreasing on $[T_2, T_{3k}]$; in particular

$$L \leq y(T_{3k}; m_1) < y(T_{3k}; m_k) \leq y(t; m_k) \leq y(T_2; m_k) \leq p_0 \quad (10)$$

for $t \in [T_2, T_{3k}]$.
The observation employed by Lasota and Opial is

$$0 > \frac{y(T_{3k}; m_k) - y(T_2; m_k)}{T_{3k} - T_2} \geq \frac{L - p_0}{T_{3k} - T_2} \geq \frac{L - p_0}{T_3 - T_2} = K_1. \quad (11)$$

Apply the mean value theorem (or (6) in the case $i = 1$) to the left hand side of (11), to see that

$$S_{k1} = \{t \in [T_2, T_{3k}] : K_1 - 1 \leq y'(t; m_k) < 0\} \neq \emptyset;$$

by the continuity of $y'(t; m_k)$, there exists a closed interval of positive length,

$$I_1 = [T_{2k1}, T_{3k1}] \subset S_{k1} \subset [T_2, T_{3k}].$$

To outline an induction argument in i, the order of the derivative $y^{(i-1)}$, set $h = \frac{T_{3k1} - T_{2k1}}{2}$ and consider

$$\frac{y(T_{2k1}; m_k) - 2y(T_{2k1} + h; m_k) + y(T_{2k1} + 2h; m_k)}{h^2}.$$

Then, continuing to observe that $y(t, m_k)$ is decreasing on I_1,

$$\begin{aligned}\frac{y(T_{21}; m_k) - 2y(T_{21} + h) + y(T_1 + 2h)}{h^2} &\geq \frac{2(L - p_0)}{h^2} \\ &= \frac{2^3(L - p_0)}{(T_{3k1} - T_{2k1})^2} \geq \frac{2^3(L - p_0)}{(T_3 - T_2)^2} = K_2\end{aligned}$$

and

$$\frac{y(T_{21}; m_k) - 2y(T_{21} + h) + y(T_1 + 2h)}{h^2} \leq \frac{2(p_0 - L)}{h^2} \leq -K_2.$$

In particular,

$$\left|\frac{y(T_{21};m_k)-2y(T_{21}+h)+z(T_1+2h)}{h^2}\right|\leq -K_2.$$

Apply (6) in the case $i=2$ and the set

$$S_{k2}=\{t\in[T_{2k1},T_{3k1}]:|y''(t;m_k)|\leq -K_2+1\}\neq\emptyset$$

and contains a closed interval of positive length

$$I_2=[T_{2k2},T_{3k2}]\subset S_{k2}\subset[T_{2k1},T_{3k1}]\subset[T_2,T_3].$$

The induction hypothesis is then, for $i \in \{2, \dots n-2\}$ assume there exist $T_{2ki} < T_{3ki}$ such that $I_i = [T_{2ki}, T_{3ki}] \subset [T_{2k(i-1)}, T_{3k(i-1)}] \subset [T_2, T_3]$ and

$$|y^{(i)}(t; m_k)| \leq -K_i + 1, \quad t \in I_i$$

where

$$K_i = \frac{i^i 2^{i-1}(L - p_0)}{(T_3 - T_2)^i}.$$

Set $h = \frac{T_{3ki} - T_{2ki}}{i+1}$. Then,

$$\begin{aligned}
\Big|\frac{\sum_{l=0}^{i+1}(-1)^{i+1-l}\binom{i+1}{l} y(T_{2ki} + lh)}{h^{i+1}}\Big| &\geq \frac{(i+1)^{i+1} 2^i (L - p_0)}{(T_{3ki} - T_{2ki})^{i+1}} \\
&\geq \frac{(i+1)^{i+1} 2^i (L - p_0)}{(T_3 - T_2)^{i+1}} = -K_{i+1}.
\end{aligned}$$

Apply (6) in the case $i+1$ and the set,

$$S_{k(i+1)} = \{t \in [T_{2ki}, T_{3ki}] : |y^{(i+1)}(t; m_k)| \leq -K_{i+1} + 1\} \neq \emptyset$$

and contains a closed interval of positive length

$$I_{i+1} = [T_{2(i+1)}, T_{3(i+1)}] \subset [T_{2i}, T_{3i}] \subset [T_2, T_3].$$

Recall that k is fixed. For this fixed k, choose $t_k \in I_{n-1}$. Then

$$
\begin{aligned}
&(t_k, y(t_k; m_k), y'(t_k; m_k), \ldots, y^{(n-1)}(t_k; m_k)) \\
&\qquad\qquad \in [T_2, T_3] \times [L, p_0] \times \Pi_{i=1}^{n-1}[-K_i - 1, K_i + 1].
\end{aligned}
$$

The set on the righthand side is a compact subset of $\mathbb{R}^{n+1}$ and independent of k. Perform this process for each k and generate a sequence

$$
\begin{aligned}
&\{(t_k, y(t_k; m_k), y'(t_k; m_k), \ldots, y^{(n-1)}(t_k; m_k))\}_{k=1}^{\infty} \\
&\qquad\qquad \subset [T_2, T_3] \times [L, p_0] \times \Pi_{i=1}^{n-1}[-K_i - 1, K_i + 1].
\end{aligned}
$$

In particular, there exists a convergent subsequence (relabeling if necessary)

$$\{(t_k, y(t_k; m_k), y'(t_k; m_k), \ldots, y^{(n-1)}(t_k; m_k))\} \to (t_0, c_1, \ldots, c_n)$$

where $t_0 \in [T_2, T_3]$. Since $t_0 \in (a, b)$ and by the continuous dependence of solutions of initial value problems, $y(t; m_k)$ converges in $C^{n-1}[T_1, T_3]$ to a solution, say $z(t)$, of the initial value problem (1), with initial conditions, $y^{(i-1)}(t_0) = c_i$, $i = 1, \ldots, n$. Thus, $p_0 = z(T_2)$ which implies $p_0 \in \Omega$ and Ω is closed. This completes the proof if $y'(T_2; m_k) < 0$ for each k.

If $y'(T_2; m_k) > 0$ for each k, find $T_1 < T_3 < T_2$ such that $y'(t; m_1) \geq 0$, for $t \in [T_3, T_2]$. Then

$$L = y(T_3; m_1) < y(T_3; m_k) \leq y(t; m_k) \leq p_0, \quad T_3 \leq t \leq T_2,$$

and the above argument can be modified to apply on $[T_3, T_2]$. This completes the proof.

Corollary

Assume that $f : (a, b) \times \mathbb{R} \to \mathbb{R}$ *is continuous and assume there exists a positive constant,* P, *such that*

$$|f(t, y) - f(t, z)| \leq P|y - z|$$

for all $(t, y), (t, z) \in (a, b) \times \mathbb{R}$. *Assume in addition, that* $\frac{\partial f}{\partial y} = f_y : (a, b) \times \mathbb{R} \to (0, \infty)$. *Then for each* $a < T_1 < T_2 < b$, $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, *the two point boundary value problem* (2), (3) *has a solution.*